Sylvester-Gallai Theorem in Incidence Geometry

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Ross Mathematics Program Lecture 1

Abstract

This is the first talk of a lecture series on geometry theorems. The first talk of this series explores the types of questions that define Incidence Geometry, focusing on the famous Sylvester-Gallai Theorem and the intriguing questions (including open ones) that stem directly from it. I also introduce projective spaces, a useful tool for many areas of geometry, and demonstrate how they can be beautifully put into action to solve problems related to the Sylvester-Gallai Theorem. We also look at progress made on defining bounds related to the theorem using projective space. To conclude, I explain the principle of duality, and we revisit the Sylvester-Gallai Theorem, but now in its dual form. The concept of duality in a real projective plane is one everybody should explore at least once in their life.

Introduction to Incidence Geometry

Incidence geometry explores the properties of incidence structures, which consist of two fundamental elements: points and lines. The core focus is on their mutual relationship, known as incidence. Unlike other branches of geometry, incidence geometry disregards factors such as angles, continuity, and length, as well as the concept of betweenness. In this context, the sequence of points on a line is irrelevant; the sole concern is whether or not a point lies on a given line.

Incidence

Incidence of a Point and a Line: A point p is said to be incident with a line l if $p \in l$.

Other types of incidences also exist; they include point-line, point-plane incidence, line circle incidence, and line surface incidence.

Introduction to Silvester-Gallai Theorem

Every finite set of points in the plane naturally defines a set of lines, the connecting lines.

Connecting Line

A connecting line is a line passing through at least two points of the set.

The Silvester-Gallai theorem is a theorem concerned with a specific type of connecting line called an ordinary line.

Ordinary Line

In combinatorial geometry, an ordinary line for a given finite point set P is defined as a connecting line that passes through exactly two points of P (a line with two point-line incidences).

Notation

Let $ol(n)$ denote the minimum number of ordinary lines determined by n non-collinear points in the plane.

Let's examine the incidence relationships between points and lines in a typical example of an incidence geometry theorem, exploring how a set of points in a plane must give rise to specific connecting lines and, more precisely, specific ordinary lines.

Imagine 3 points in a plane. Can we always draw an ordinary line? If they aren't co-linear, obviously, yes. If they are co-linear, obviously, no.

What if we add more points? For n points, we can think of a construction where there are only 3 ordinary lines. Since there are no constructions out of 3 points with less ordinary lines, ol(4)=3. From now on, we will always exclude the case of co-linear points because there will clearly never be any ordinary lines in that case, as the one line we can draw through the points will have too many points on it.

If we have *four* non-co-linear points, it is also easy to see that we can draw an ordinary line no matter how we rearrange them.

What if we add a 5th point to the previous drawing? It seems like, still, there are ordinary lines.

What about for 6 points?

What about 7 points?

It seems completely trivial that there are always ordinary lines for small n, but what about when n gets larger?

This question was originally posed by James Joseph Sylvester in 1893. His specific words are as follows:

"Is it true that any finite set of points in the plane, not all on a single line, contains two points such that their connecting line does not pass through a third point?"

Mathematicians worked on the problem for many decades. Then, finally, 40 years later, Tibor Grünwald (Gallai) proved that the answer was yes with a 3-line proof, shocking the math community.

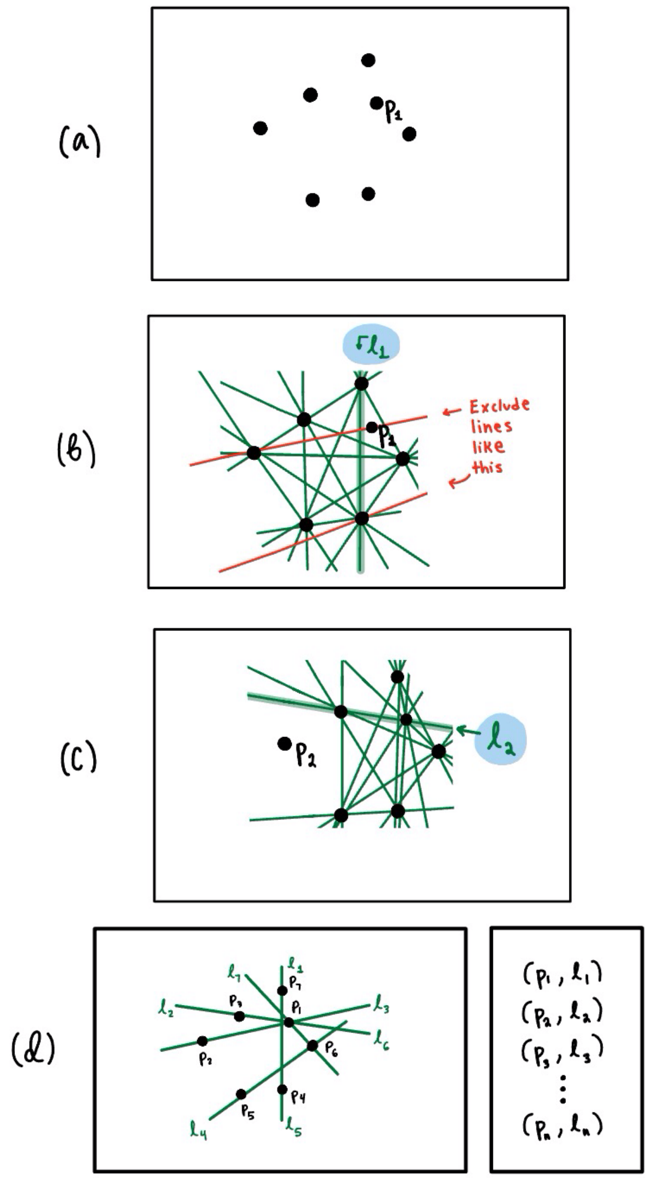
Sylvester-Gallai Theorem

For any finite set of points in the plane, not all of which are collinear, there exists an ordinary line.

Proof of Sylvester-Gallai Theorem

Take a set of points P in $\mathbb{R}^2$ consisting of n points. Choose one point p_1 out of our set of points, as shown in Fig (a) below. Next, consider all lines going through at least 2 points and not going through the chosen point. Out of all lines, we choose the line with the smallest distance from p_1 and call it l_1. This is depicted in Fig (b).

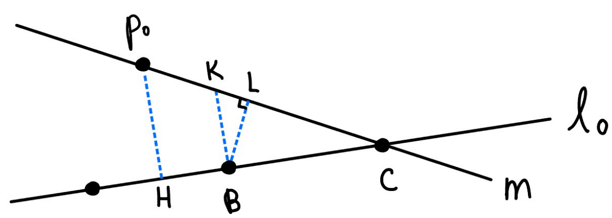
Take another point, p_2, and look at all lines going through at least 2 points in the set and not going through p_2. Find the closest such line to p_2 and call it l_2, as shown in Fig (c). Repeating this process for all points in our set P results in n "point-line" pairs since n is the number of points in our set.



For each pair, we calculate the distance from the point to the corresponding line and choose the pair where that distance is the smallest. Call this point and this line (p_0, l_0). We chose this pair out of a set where the lines went through at least 2 points, but it ends up that this pair l_0 must go through *exactly* 2 points in the set. Let's see why.

Suppose that this is not true. In other words, there are at least 3 points on the line l_0. Draw a line from p_0 perpendicular to l_0 and call the point where the lines meet H. The point H splits l_0 into two rays. At least on one ray, there are at least 2 points. This does not exclude the case where one of the points coincides with H.

Look at the ray on which two points are located. Denote the closest of the points B and the second closest point C. Draw a line between p_0 and C and call it m. Next, draw a line from B perpendicular to l_0 and intersecting m at point K. Next, draw a line from B, which is perpendicular to m, and call the point where the lines intersect L. It is clear that the length of the segment BK is strictly greater than the length of the segment BL, but less than or equal to the length of the segment p_0H. Hence, $BL < p_0H$. The line m passes through at least two points of the set and does not pass through point B. But this fact contradicts the fact that the distance (p_0, l_0) is the minimum distance between a point and a line. This contradiction signifies that our assumption that l_0 contains more than 2 points isn't true, and so it must pass exactly 2 points. Hence, the Sylvester-Gallai theorem is proven.



$$BL < BK \leq p_0 H$$
$$BL < d(p_0, l_0)$$
$$d(B, m) < d(p_0, l_0) \quad \Rightarrow\Leftarrow$$

Lower Bound on $ol(n)$

Now that we know that there must exist at least 1 ordinary line for a finite set of non-colinear points and lines, ($ol(n) \geq 1$), it is logical to wonder if the lower bound is higher. It is commonly believed that the lower bound *is* in fact higher, and a fundamental question in this field is determining this lower bound. Dirac and Motzkin made a conjecture for a possible such bound.

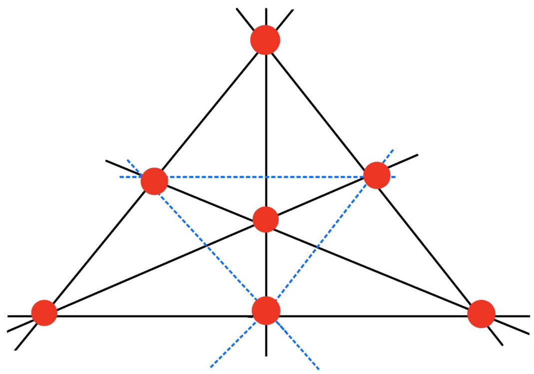
Dirac-Motzkin Conjecture:

For every $n \neq 7, 13$, we have

$$ol(n) \geq \lceil \frac{n}{2} \rceil$$

Remember, n is the number of vertices in a configuration Q.

In the image below, we see that the number of ordinary lines is three, which is less than $\lceil \frac{7}{2} \rceil$. This example is why the conjecture excludes $n = 7$.



It is also interesting to explore the upper bound of $ol(n)$, which will require some understanding of projective planes.

Projective Planes

Intuitively speaking, a projective plane is like the real plane, except we also include points called "points at infinity." The projective plane is a completion of the Euclidean plane.

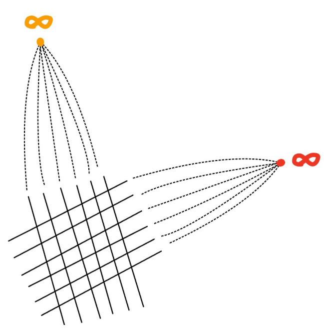
Point at Infinity

In geometry, a point at infinity is an idealized limiting point at the "end" of each line. It is used in projective geometry to complete the Euclidean Space.

Imagine looking at two railroad tracks going into the distance. The point where they seem to meet is called the "point at infinity."

Take a line in $\mathbb{R}^1$. Now add the point at infinity as shown below, and now we have a line in projective line. It may seem like I added 2 points, but actually, I added only one because these are really the same point: the point at infinity of this line. It may be easier to think about our line as a circle so the point at infinity resembles a singular point in this representation rather than two separate ones (there are also other definitions, which we will not use).

In $\mathbb{RP}^2$, we identify classes of lines by whether they are parallel in the Euclidean plane and say that each class of lines intersects in one singular point at infinity, as shown below.



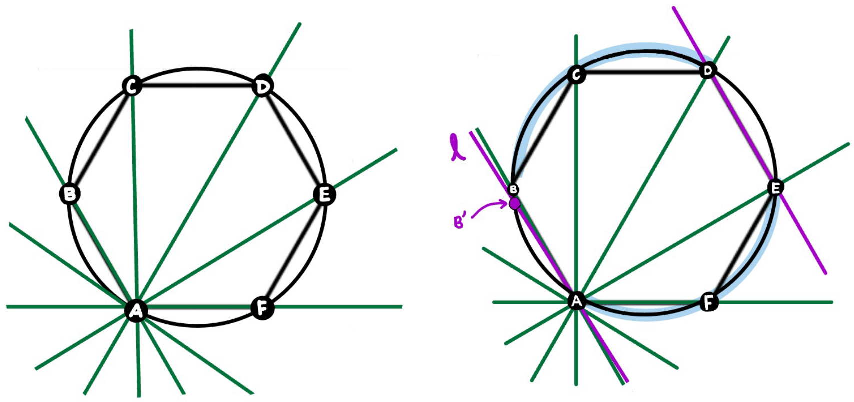
When you extend a line indefinitely in both directions, it intersects the "line at infinity" at a unique point. All classes of parallel lines intersect at the same point at infinity, but different classes intersect the line at infinity at distinct points. This concept allows for a more comprehensive and symmetrical treatment of geometric properties and simplifies many theorems.

You can think of a point at infinity as the point where train tracks (parallel lines) appear to meet, and the horizon as the line at infinity.

Now that we know what projective spaces are, it would be useful to know how to make configurations where we have specific numbers of points at infinity. The following lemma helps up build beautiful constructions.

Lemma: A set of connecting lines on a regular polygon with n/2 vertices can naturally define n points total

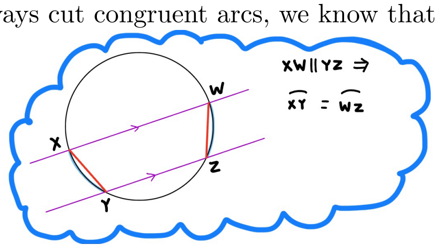
Using the fact that any regular polygon can be inscribed in a circle, we can inscribe our regular polygon with n/2 vertices into a circle. Call one vertex A. Next, draw a tangent to the circle there, and draw a line from A to all of the other vertices of the shape. The picture below on the left illustrates this process for a hexagon.



This should define all of the directions that can be drawn by connecting two vertices of this shape for the following reason:

We want to show that if we pass a line BC through two points on our hexagon (two points disjoint from A), then the line parallel to it that goes through A will be in one of the directions we already drew.

Let l be the line parallel to BC that goes through point A, as shown in the picture above on the right. Using the fact that parallel chords in the same circle always cut congruent arcs, we know that arcAE = arcB'D.



Since the shape is regular, the length from around the circumference from A to E is the same as the distance between any 2 vertices. So, arcBD starting from B goes a length of 2 the distance between 2 veriticies and the point B' should coincide with B. Since the line going between any two points is unique, we see that the line l is actually a direction we already had.

Upper Bound on $ol(n)$

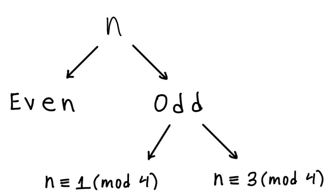
It is easy to check that $ol(3) = 3$, $ol(4) = 3$, $ol(5) = 4$.

Upper Bound Theorem

For even $n \geq 6$, we have $ol(n) \leq \frac{n}{2}$. For odd $n \geq 6$, we have $ol(n) \leq \lfloor 3\frac{n}{4} \rfloor$.

Recall, the $ol(n)$ notation denotes the *minimal* number of ordinary lines for number of points, not simply the number of ordinary lines. As long as we show there is always some configuration of n points where $ol(n)$ equals the upper bound, we have a proof. Maybe in some other configuration, we can have less ordinary lines, but that is irrelevant because we are curious about the maximum number. Maybe some other configuration has more ordinary lines, but if we show that it is possible to have a configuration where the number of ordinary lines falls within our bounds, then this is irrelevant because it's not the *minimal* number of ordinary lines.

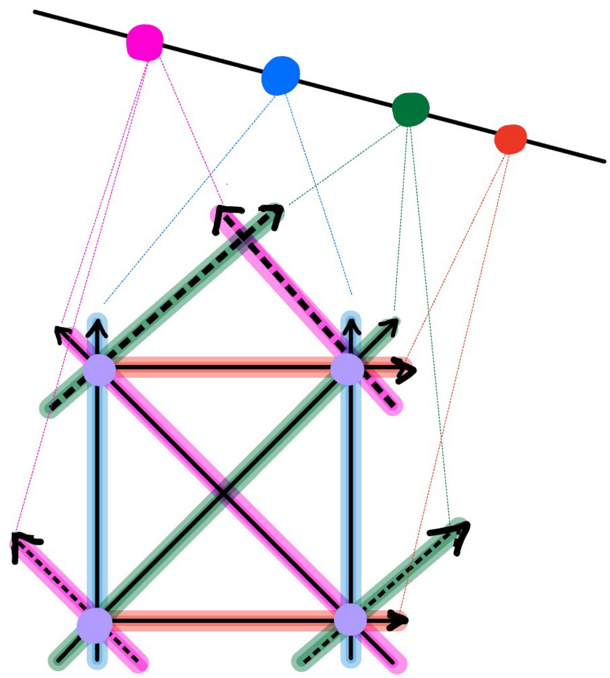
We split the possible n values into 3 cases,



Case 1: n is even

Constructing the perfect shape:

For our example of n being even, consider a configuration Q with four vertices (finite points) in $\mathbb{PR}^2$. In the picture below, we have 4 directions, which results in 4 points at infinity. Summing up the finite points and the four points at infinity, we see that the configuration in $\mathbb{PR}^2$ is a shape with n=8 points.



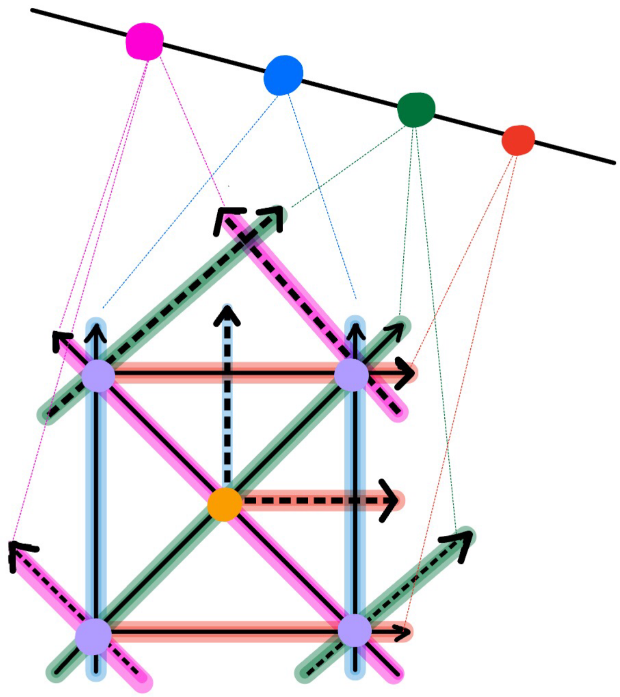
Let's take a look at why, in the general case, a regular polygon with n/2 vertices has n/2 directions, which defines n/2 points at infinity. So, to get a shape with n points, where n is even, we can draw a regular n/2-gon (regular polygon with n/2 vertices).

□

Now, we know how to build a shape with the number of points we want. The next step is to count the number of ordinary lines. The key observation is that the ordinary lines are the ones that are tangent to the vertices. This is easy to see by looking at the picture above. so the number of ordinary lines is the number of vertices. Since we have $n/2$ vertices, we get $n/2$ ordinary lines.

Example of Case 2 $n \equiv 1(mod4)$ **Using n=9**

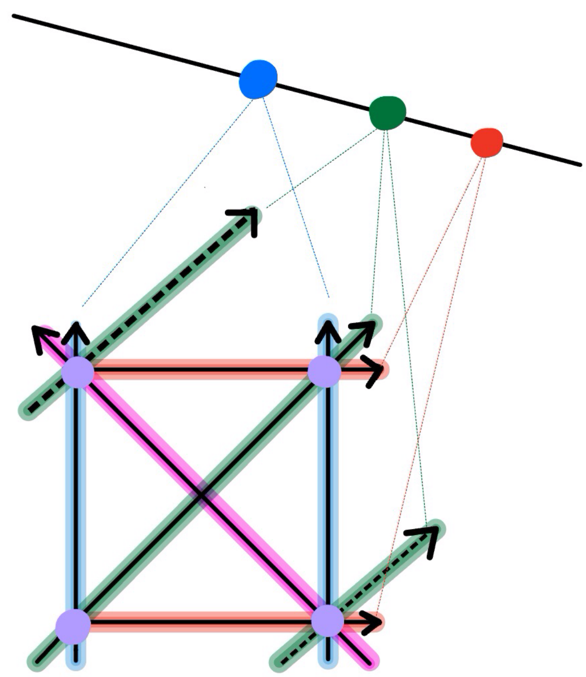
My first goal is to create a 9-point construction. To do this, I take the 8-point shape from before and add 1 point in the center as you can see below.



The new shape includes all of the directions we had previously. More generally, for an $n \equiv 1(mod4)$, $n = 4k + 1$ for some $k \in \mathbb{Z}$, so in that case we start with a regular polygon of $\frac{n-1}{2} = 2k$ vertices where the number of vertices is even. It is important that the number of vertices is even because otherwise the big diagonals won't go through the center of the shape. An even-gon with 2k vertices has 2k directions. By adding the new point, we add k ordinary lines. The total number of ordinary lines is the sum of all the old ordinary lines and the new ordinary lines, so for $n \equiv 1(mod4)$, there are k + 2k = 3k total directions. This is the same thing as the problem statement because $3\lfloor \frac{n}{4} \rfloor = 3\lfloor \frac{4k+1}{4} \rfloor = 3\lfloor k + \frac{1}{4} \rfloor = 3k$.

Example of Case 3 $n \equiv 3(mod4)$ **Using n=7**

The construction of a shape with n=7 is as follows. Take the construction on n+1 points (8 points in this case) and remove 1 point at infinity.



Since (n-1)/2 is even, all of the ordinary lines are tangent to a vertex because ordinary lines needed to consist of one finite point and one infinitely distant point. By taking away this point at infinity, some of the lines that were previously ordinary became just normal lines going through one point because they lost the second point, the point at infinity. There will be precisely 2 such ordinary lines lost because each direction is defined by a pair of lines tangent to vertices on opposite sides. Removing a point at infinity also creates new ordinary lines because now lines that consist of two finite points can potentially be ordinary lines since they don't have to have another infinitely distant point on them. How many ordinary lines can we draw on our new shape consisting of just 2 finite points? The answer is that we can draw $\frac{\frac{n+1}{2}-2}{2}$ such lines because such a diagonal can be drawn between pairs of points parallel to that direction on opposite sides of the shape.

So, by removing the point from our configuration with an even number of points, we lose 2 ordinary lines and gain $\frac{n-1}{2} - 2$ ordinary lines. If we let $n = 4k + 3$ for some $k \in Z$, then in total, the number of ordinary lines is

$$\frac{n+1}{2} - 2 + \frac{\frac{n+1}{2} - 2}{2} =$$

$$\frac{4k+4}{2} - 2 + \frac{\frac{4k+4}{2} - 2}{2} =$$

$$= 2k + 2 - 2 + \frac{2k + 2 - 2}{2} =$$
$$2k + k = 3k$$

As we saw earlier, the theorem's bound is equivalent.

So, we have looked at all cases, and have seen that the claim is always satisfied.

Duality in Real Projective Plane

Duality in Real Projective Plane

For every configuration S of points and lines in $\mathbb{RP}^2$ we can find a dual configuration S^* in $\mathbb{RP}^2$ with the following properties.

*Every point in S corresponds to one line in S^* and vice versa.

*Every line in S corresponds to one point in S^* and vise versa.

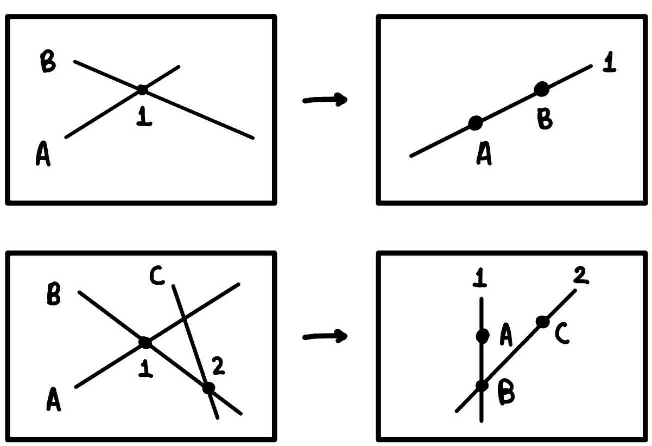
*A point and a line in S are incident if and only if the corresponding line and point in S^* are incident.

*A set of points in S is collinear if and only if the corresponding lines in S^* are concurrent.

*A set of lines in S is concurrent if and only if the corresponding points in S^* are collinear.

(Concurrent lines are three or more lines that intersect at a single point.)

Let's look at some examples.



Sylvester-Gallai theorem – dual version

Using the property of duality in the real projective plane we get a new theorem.

Sylvester-Gallai theorem – dual version

Every arrangement of finitely many lines in $\mathbb{R}^2$, not all concurrent, and not all parallel, admits an ordinary point.

At first glance, it may seem like this theorem is quite different from the original. First of all, the original theorem statement talks about $\mathbb{RP}^2$ whereas this on is in $\mathbb{R}^2$. However, this actually doesn't matter for the following reason. Since we are working with a finite number of points and lines, we can always adjust a pair of parallel lines slightly in our construction so that they are no longer parallel, moving a previous point at infinity to $\mathbb{R}^2$, and not changing the number of other incidences in our set. Using the property of duality in the projective plane, we do not have to prove the statement. Instead we just know it is true because the duality of a statement in projective planes is true.